30 Applications of the AdS/CFT Correspondence

30.1 Static Quark Potential

If we we stack $N+1$ D3-branes in order to get an $SU(N+1)$ $\mathcal{N}=4$ gauge theory and then pull one of the branes a distance u away, we break the gauge symmetry to $SU(N)$ and the stretched string states correspond to massive gauge bosons with mass

$$m_W = \frac{u}{\alpha'} \ . \tag{30.1}$$

These states transform as $\Box$ and $\overline{\Box}$ of $SU(N)$. In the limit $u \to \infty$ such a string behaves like an infinitely heavy (static) quark. If we have a static quark and antiquark pair separated by a distance L on the boundary of AdS_5, the supergravity solution that minimizes the action is one where the string stretches from the quark to the antiquark along a geodesic.

We can thus calculate the expectation value of a Wilson Loop in this theory. For a infinitely heavy quark which traverses a closed curve C the Wilson loop is

$$W(C) = \mathrm{Tr}\, P \exp i \int_C A \tag{30.2}$$

The AdS/CFT correspondence gives us that

$$\langle W(C) \rangle = e^{-\alpha(D)} \tag{30.3}$$

where D is the surface of minimal area in AdS_5 that has C as its boundary, and $\alpha(D)$ is a regularized area of D. The surface D corresponds to the string stretched between the quarks. We are allowed to subtract a term proportional to the circumference of C, which corresponds to the action of the widely separated heavy quarks. If C is a a square in Euclidean space of width L and height T, then the expectation value of the Wilson loop gives us the potential energy of the quark-antiquark pair:

$$\langle W(C) \rangle = e^{-TV(L)} \tag{30.4}$$

Using the metric

$$ds^2 = \frac{R^2}{z^2}(dz^2 + \sum_{i=1}^{4} dx^i dx^i) \tag{30.5}$$

we see that as we scale the size of C by

$$x_i \to \rho\, x_i \tag{30.6}$$

we can keep $\alpha(D)$ fixed by scaling D:

$$\begin{aligned} x_i &\to \rho\, x_i && (30.7)\\ z &\to \rho\, z\ , && (30.8) \end{aligned}$$

so $\alpha(D)$ is independent of ρ, or in other words $\alpha(D)$ is not proportional to Area$(C) \sim \rho^2$. Extracting the potential one finds

$$V(L) \sim -\frac{\sqrt{g^2 N}}{L} \tag{30.9}$$

the $1/L$ behavior is required by conformal symmetry, while the $\sqrt{g^2 N}$ behavior is different from, but not in contradiction with, the perturbative result.

30.2 Breaking SUSY: Finite Temperature

As is well known, if we take Euclidean time ($t_E = -it$) to be periodic:

$$t_E \sim t_E + \beta \tag{30.10}$$

then

$$e^{itE} \to e^{-\beta E} \tag{30.11}$$

and we can get finite temperature 4 dimensional gauge theory. To do this we must impose periodic boundary conditions on the bosons and anti-periodic boundary conditions on the fermions. This leaves some zero-energy boson modes, but no zero-energy fermion modes, so SUSY is broken. Scalars will get masses from loop effects (gauge mediation) while gluons are protected by gauge symmetry. So the low-energy effective theory is pure non-supersymmetric Yang-Mills. In the high temperature limit we get a zero-temperature 3 dimensional gauge theory

In AdS Hawking and Page showed long ago that there is a phase transition. In the high temperature limit the partition function is dominated by a black-hole metric with a horizon size proportional to the temperature.

The metric for black hole on AdS_5 is:

$$\frac{ds^2}{R^2} = \left(u^2 - \frac{b^4}{u^2}\right)^{-1} du^2 + \left(u^2 - \frac{b^4}{u^2}\right) d\tau^2 + u^2 dx^i i dx^i \tag{30.12}$$

horizon size is $b = \pi T$. We can now check that the AdS/CFT correspondence is in accord with our knowledge about non-supersymmetric non-Abelian gauge theories. The main thing that we know about such theories is that they confine.

30.3 Wilson Loops at High Temperature

Calculating the Wilson loop expectation value

$$\langle W(C) \rangle = e^{-\alpha(D)} \tag{30.13}$$

with the black hole metric (30.12) we note that u is bounded by the horizon b, so the minimal area of D is just its area at the horizon

$$\alpha(D) = R^2 b^2 \, \text{Area}(C) \tag{30.14}$$

which corresponds to Area Law confinement, or a linear potential

$$V(L) = R^2 b^2 L \ . \tag{30.15}$$

Note that the string tension is very large:

$$\sigma \sim R^2 b^2 \sqrt{g^2 N} \tag{30.16}$$

30.4 The Glueball Mass Gap

We also know that confining gauge theories should have a mass gap. To see this recall that there is a massless scalar field Φ in AdS_5 (the dilaton) which couples to Tr F^2, and Tr F^2 has a non-zero overlap with states with $J^{PC} = 0^{++}$. So calculating the two point function of this operator will give us information about the mass of the 0^{++} glueball.

Inserting the black-hole metric (30.12) into the wave equation

$$\partial_\mu \left[\sqrt{g} g^{\mu\nu} \partial_\nu \Phi \right] = 0 \tag{30.17}$$

and looking for a plane wave solution on the boundary

$$\Phi = f(u) e^{ik.x} \tag{30.18}$$

we find (we have set the momenta in S^5 to zero, since non-zero values correspond to heavier Kaluza-Klein (KK) modes)

$$u^{-1}\frac{d}{du}\left(\left(u^4-b^4\right)u\frac{df}{du}\right)-k^2f=0 \tag{30.19}$$

For large u we have $f\sim u^\lambda$ where $m^2=0=\lambda(\lambda+4)$ so as $u\to\infty$ either $f\sim$ constant or $f\sim u^{-4}$. Only the second solution gives a normalizable solution (and hence a finite action). We also need f must be regular at $u=b$, which implies $\frac{df}{du}$ is finite. This is essentially a wave guide problem. There are no normalizable solutions for $k^2\geq 0$, and there are discrete eigenvalues solutions for $k^2<0$ The glueball masses are given by:

$$M_i^2=-k_i^2>0 \tag{30.20}$$

as expected.

We can also get glueball masses in 4D, starting with D4 branes. The problem is that the supergravity limit $g\to 0$, $g^2N\to\infty$ does not correspond to the gauge theories we usually think about. We can see this by considering the intrinsic scale where the non-supersymmetric gauge theory becomes strong compared to the effective cutoff where extra particle thresholds appear (T). For QCD_3 the intrinsic scale is given by the gauge coupling:

$$g_3^2N=g^2NT \tag{30.21}$$

to keep this fixed as $T\to\infty$ we need to take $g^2N\to 0$ Similarly in QCD_4

$$\Lambda_{\mathrm{QCD}}=\exp\left(\frac{-24\pi^2}{11\,g^2N}\right)T \tag{30.22}$$

to keep this fixed as $T\to\infty$ we need to take $g^2N\to 0$.

So we can only do the calculation where the extra states have similar masses to the glueballs we want. Things can be improved a little by considering rotating branes, there the KK modes associated with the compact Euclidean time can be removed. Denoting the angular momentum of the branes in the extra (S^5) directions by a, and taking the large a limit. One finds surprisingly good values for the masses. It is not known whether this agreement is coincidental or has an underlying reason.

state	lattice $N = 3$	SUGRA $a = 0$	SUGRA $a \to \infty$
0^{++}	1.61 ± 0.15	1.61 (input)	1.61 (input)
0^{++*}	2.48 ± 0.23	2.55	2.56
0^{-+}	2.59 $\pm$0.13	2.00	2.56
0^{-+*}	3.64 $\pm$0.18	2.98	3.49

30.5 Breaking SUSY with Orbifolds

At large N we have:

$$
\begin{array}{lcr}
\text{Type IIB on } \mathrm{AdS}_5 \times S^5 & \Leftrightarrow & \mathcal{N} = 4\,\mathrm{CFT} \\
\mathrm{KK-mode} & & \text{operator} \\
& & \\
\downarrow & \text{“orbifolding” } S^5 & \downarrow \\
& & \\
\mathrm{AdS}_5 \times S^5/G & \Leftrightarrow & \mathcal{N} < 4\,\mathrm{CFT} \\
\text{invariant}\,\mathrm{KK-mode} & & \text{invariant operator}
\end{array}
$$

It is well known that the large N limit of the gauge theory is dominated by planar diagrams. It has been shown that the planar diagrams of certain orbifolded gauge theories that the planar diagrams are proportional to the planar diagrams of the full theory up to a rescaling of the gauge coupling. So in the large N limit these theories are also conformal. By orbifolding in different ways we can break different amounts of SUSY:

$$
\begin{array}{lll}
SU(4) \supset G, & SU(3) \not\supset G & \Rightarrow \mathcal{N} = 0 \\
SU(3) \supset G, & SU(2) \not\supset G & \Rightarrow \mathcal{N} = 1 \\
SU(2) \supset G, & & \Rightarrow \mathcal{N} = 0
\end{array}
$$

The simplest case is $G = Z_k$. We also need to specify how G is embedded in the gauge group. To do this we need the regular representation of Z_k:

$$\gamma^a = \mathrm{diag}(\omega^0, \omega^a, \omega^{2a}, \ldots, \omega^{(k-1)a}) \tag{30.23}$$

where

$$\omega = e^{2\pi i/k} \tag{30.24}$$

$$a = 0, 1, \ldots, k-1 \tag{30.25}$$

We can embed Z_k in $SU(kN)$ by defining

$$\gamma_N^a = \mathrm{diag}(\mathbf{1_N}, \mathbf{1_N}\omega^a, \mathbf{1_N}\omega^{2a}, \ldots, \mathbf{1_N}\omega^{(k-1)a}) \tag{30.26}$$

so that the adjoint transforms as

$$Ad \to \gamma_N^a Ad (\gamma_N^a)^\dagger \tag{30.27}$$

The invariant pieces are k adjoints of k different $SU(N)$'s. The orbifolded theories constructed using the regular representation in this fashion are the set that remain conformal.

Take as example the Z_6 orbifold where the embedding of Z_6 in the global $SU(4)$ is such that the four fermion fields transform as:

$$\psi \to (\omega^a, \omega^{2a}, \omega^{3a}, \omega^{4a})\psi \tag{30.28}$$

The gauge field transforms as

$$A \to \begin{array}{c|cccccc} & \omega^{0*} & \omega^{a*} & \omega^{2a*} & \omega^{3a*} & \omega^{4a*} & \omega^{5a*} \\ \omega^0 & I & & & & & \\ \omega^a & & I & & & & \\ \omega^{2a} & & & I & & & \\ \omega^{3a} & & & & I & & \\ \omega^{4a} & & & & & I & \\ \omega^{5a} & & & & & & I \end{array} \tag{30.29}$$

where I denotes the invariant sub-block. Consider the fermion component that transforms as

$$\psi \to \omega^{2a} \begin{array}{c|cccccc} & \omega^{0*} & \omega^{a*} & \omega^{2a*} & \omega^{3a*} & \omega^{4a*} & \omega^{5a*} \\ \omega^0 & & & I & & & \\ \omega^a & & & & I & & \\ \omega^{2a} & & & & & I & \\ \omega^{3a} & & & & & & I \\ \omega^{4a} & I & & & & & \\ \omega^{5a} & & I & & & & \end{array} \tag{30.30}$$

The invariant fermions are obviously bifundamentals. The content of the orbifolded theories can be summarized in a "moose" or "quiver" diagram.

Frampton and Vafa proposed that such theories could solve the hierarchy problem if physics was conformal above 1 TeV, since an exactly conformal theory has no quadratic divergences. However if we consider the effective theory below some scale μ and we can calculate the one-loop β functions (using bird track notation is the easiest way to do it) and then set the β functions to zero. One finds fixed points at

$$
\begin{aligned}
Y_* &= g\sqrt{1-\frac{1}{N^2}} && (30.31)\\
\lambda_{i*} &= g^2(1+\frac{a_i}{N^2}+\frac{b_i}{N^4}+\ldots);\ i=1,\ldots,5 && (30.32)\\
& && (30.33)
\end{aligned}
$$

and the one-loop scalar mass given by

$$
m_\phi^2 = \left[Nc_i\lambda_i + 3\frac{N^2-1}{N}g^2 - 8NY^2\right]\frac{\mu^2}{16\pi^2} \tag{30.34}
$$

IN the large N limit $\sum_i c_i = 5$ and we recover the $calN = 4$ result that there is no quadratic divergence. To leading order in N we have:

$$
m_\phi^2 = \frac{3g^2}{N}\frac{\mu^2}{16\pi^2}\ . \tag{30.35}
$$

So to get $m_\phi = 1$ TeV, with $\mu = M_{\rm Pl}$ we need $N = 10^{28}$.

References

[1] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. **B428** (1998) 105 hep-th/9802109.

[2] E. Witten, "Anti-de Sitter space, thermal phase transition, and confinement in gauge theories," Adv. Theor. Math. Phys. **2** (1998) 505 hep-th/9803131.

[3] C. Csaki, H. Ooguri, Y. Oz and J. Terning, "Glueball mass spectrum from supergravity," JHEP **01** (1999) 017 hep-th/9806021.

[4] C. Csaki, Y. Oz, J. Russo and J. Terning, "Large N QCD from rotating branes," Phys. Rev. **D59** (1999) 065012 hep-th/9810186.

[5] S. Kachru and E. Silverstein, "4d conformal theories and strings on orbifolds," Phys. Rev. Lett. **80** (1998) 4855 hep-th/9802183.

[6] A. Lawrence, N. Nekrasov and C. Vafa, "On conformal field theories in four dimensions," Nucl. Phys. **B533** (1998) 199 hep-th/9803015.

[7] M. Bershadsky, Z. Kakushadze and C. Vafa, "String expansion as large N expansion of gauge theories," Nucl. Phys. **B523** (1998) 59 hep-th/9803076.

[8] Y. Oz and J. Terning, "Orbifolds of AdS(5) x S(5) and 4d conformal field theories," Nucl. Phys. **B532** (1998) 163 hep-th/9803167.

[9] P.H. Frampton and C. Vafa, "Conformal approach to particle phenomenology," hep-th/9903226.

[10] M. Schmaltz, "Duality of non-supersymmetric large N gauge theories," Phys. Rev. **D59** (1999) 105018 hep-th/9805218.

[11] C. Csaki, W. Skiba and J. Terning, "Beta functions of orbifold theories and the hierarchy problem," hep-th/9906057.